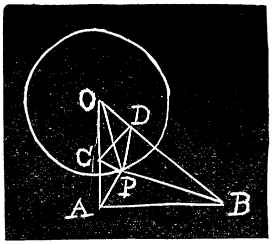



$BP$ make equal angles with the radius drawn to $P$.

This is known as Alhazen's Problem and does not admit of a solution by ruler and compasses only. However, by the use of an hyperbola an approximate solution can be effected.

*Construction.* Let $O$ be the center of the circle, $a$ its radius. Take $C$ and $D$ so that $AO.OC=a^2=BO.OD$.

Now, the locus of the vertices of the triangles whose base is $CD$ and whose base angles have a constant difference $(OCD-ODC)$ is well known to be a hyperbola. This will cut the circle in four points, of which let $P$ be one. This is the point required.

*Proof.* $\angle CDP-\angle DCP=\angle OCD-\angle ODC$ (by construction). Transposing and adding,

$\therefore \angle OCP=\angle ODP$...(1).

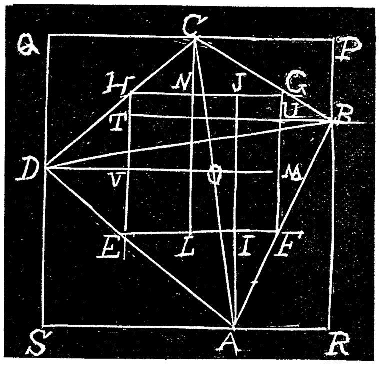
Also in the triangles $AOP$, $POC$, we have $AO{:}OP=OP{:}OC$ (by construction).

$\therefore \angle APO=\angle OCP$. Similarly, $\angle BPO=\angle ODP$ by (1).

$\therefore \angle APO=\angle BPO$, and $\therefore \angle APR=\angle BPR$. Q. E. F. Q. E. D.

**350. Proposed by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.**

Given the quadrilateral $AB=a=225$, $BC=b=153$, $CD=c=207$, $DA=d=135$, $AC=e=240$. Find the side of the square inscribed in this quadrilateral having a corner in each side.




**Solution by the PROPOSER.**

Let $ABCD$ be the given quadrilateral, $EFGH$ the inscribed square, $RPQS$ the circumscribed rectangle having its sides parallel to the sides of the square. Draw $AIJ$, $BUT$, $CNL$, $DVM$ perpendicular, respectively, to $EF$ and $HG$, $FG$ and $HE$, $HG$ and $EF$, $HE$ and $GF$.

Let $AB=a$, $BC=b$, $CD=c$, $DA=d$, $AC=e$, $BD=f$, $O$ the intersection of $AC$, $BD$, $\angle COB=\beta$, $\angle ACL=\theta=\angle CAJ$, $\angle BDM=\phi=\angle DBT$, $\angle BAC=\delta$, $\angle DAC=\gamma$, $\angle BCA=\rho$, $\angle DCA=\mu$, area $ABCD=\triangle$.

Then $\phi=\frac{1}{2}\pi-(\beta-\theta)$, $RB=a\cos(\delta-\theta)$, $RA=a\sin(\delta-\theta)$, $DS=d\cos(\gamma+\theta)$, $AS=d\sin(\gamma+\theta)$, $DQ=c\cos(\mu-\theta)$, $CQ=c\sin(\mu-\theta)$, $BP=b\cos(\rho+\theta)$, $PC=b\sin(\rho+\theta)$, $AI+x+CN=AJ+CN=e\cos\theta$, $BU+x+DV=BT+DV=f\cos\phi$.

$\therefore BT+DV=f\sin(\beta-\theta)$.

$x^2+\frac{1}{2}x(AI+BU+CN+DV)=\triangle=\frac{1}{2}x(AI+x+CN)+\frac{1}{2}x(BU+x+DV)$.

$\therefore x[e\cos\theta+f\sin(\beta-\theta)]=2\triangle$...(1).

$\triangle+\frac{1}{4}[a^2\sin2(\delta-\theta)+b^2\sin2(\rho+\theta)+c^2\sin2(\mu-\theta)+d^2\sin2(\gamma+\theta)$
$=ef\cos\theta\sin(\beta-\theta)\ldots(2).$

Reducing (2), and remembering that $4\triangle-2ef\sin\beta=0$, we get

$$\tan2\theta=\frac{a^2\sin2\delta+b^2\sin2\rho+c^2\sin2\mu+d^2\sin2\gamma-2ef\sin\beta}{a^2\cos2\delta-b^2\cos2\rho+c^2\cos2\mu-d^2\cos2\gamma-2ef\cos\beta}\ldots(3).$$

When $a=225$, $b=153$, $c=207$, $d=135$, $e=240$. Then $f=277.4$, $\triangle=30656.46$, $\delta=38^\circ\ 14'\ 54''$, $\rho=65^\circ\ 33'\ 40''$, $\gamma=59^\circ\ 24'\ 36''$, $\mu=34^\circ\ 9'\ 22''$, $\beta=67^\circ\ 3'\ 52''$.

$\therefore\tan2\theta=-.345608=2\tan\theta/(1-\tan^2\theta)$.

$\therefore\tan\theta=5.954833$ or $-0.167930$.

$\theta=80^\circ\ 28'\ 2''$ or $170^\circ\ 28'\ 2''$ and $x=-2496.97$ or $-121.044$.

Therefore, two squares can be inscribed in this quadrilateral, the smaller one truly inscribed and the larger with its corners on the sides.

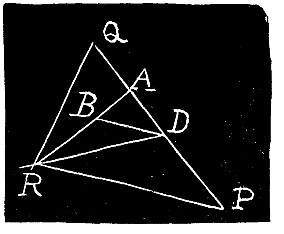
Also solved by J. Scheffer, and V. M. Spunar.

**351. Proposed by L. E. DICKSON, Ph. D., The University of Chicago.**

Given an isosceles right triangle with hypothenuse $h$; an isosceles triangle with two sides $h$ and two angles $A=22^\circ\ 30'$; a right angle triangle with the same angle $A$ and opposite side $h/\sqrt{2}$; a triangle with the same angle $A$, opposite side $h$, and an angle $45^\circ$. Form a triangle whose four pieces are these four triangles, and prove geometrically that it is isosceles.

**Solution by G. B. M. ZERR. A. M., Ph. D., Philadelphia, Pa.**

Let $ABD$ be the isosceles right triangle with hypothenuse $BD=h$, sides $AB$, $AD=h/\sqrt{2}$.




Produce $AB$ to $R$, making $BR=BD$ and connect $RD$. Since $\angle ABD=45^\circ$, $\angle RBD=135^\circ$. $\therefore\angle BRD=\angle RDB=22^\circ\ 30'=A$, and $BRD$ is the isosceles triangle with sides $h$ and opposite angles $A$. Take $Q$ in $DA$ produced so that $AQ=AD$ and join $RQ$. Then the right triangle $ARQ$ has angle $ARQ=A$ and $AQ=h/\sqrt{2}$. Produce $AD$ to $P$ making $DP=h$ and join $RP$. Triangle $DRP$ has angle $DRP=A$, $\angle P=45^\circ$, and side $DP$ opposite $\angle DRP=h$.

Now $\angle PQR=67^\circ\ 30'=3A$, $\angle ARQ=\angle ARD=\angle DRP=A$.

$\therefore\angle QRP=3A$. $\therefore\angle Q=\angle QRP$, and the triangle $PQR$ is isosceles.